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The Gray code is a mapping of the naturally ordered B-bit binary numbers to an ordering in which the binary representations of successive integers differ by exactly one bit. For example, the 4-bit Gray code representations of 7 and 8 are 0100 and 1100, respectively, which differ by one bit. The "natural" representations are 0111 and 1000, which differ by three bits. The purpose of this correspondence is to present a simple recursive method for generating the Gray code.

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A Recursive Method for Generating the Gray Code

by

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The Gray code [1] is a mapping of the naturally ordered B-bit binary numbers to an ordering in which the binary representations of successive integers differ by exactly one bit. For example, the 4-bit Gray code representations of 7 and 8 are 0100 and 1100, respectively, which differ by one bit. The "natural" representations are 0111 and 1000, which differ by three bits. The purpose of this correspondence is to present a simple recursive method for generating the Gray code. First, we briefly discuss how the need for the Gray code arises.

We consider an M-ary communication system in which one of $M=2^B$ symbols, representing B bits, is transmitted. A symbol error occurs when the transmitted symbol is S_k and S_m, $m \neq k$, is selected as the received symbol. In most systems, each of the M symbols is equally likely to be transmitted and if a symbol error occurs it is equally likely that any one of the $M-1$ incorrect symbols will be selected. As a result, the probability of a bit error P_b is given by

$$P_b = P_S \frac{2^{B-1}}{2^B - 1},$$

where P_S is the probability of a symbol error [1]. However, in systems that use pulse amplitude modulation (PAM), the most likely errors involve selection of a symbol with amplitude adjacent to the transmitted amplitude. Similarly, in systems that use pulse position modulation (PPM) or cyclic code shift keying (CCSK), the most likely errors involve selection of a symbol with time or cyclic shift adjacent to the transmitted time or cyclic shift. For these cases, the optimum indexing of the symbols is achieved by using Gray encoding.

We assume that each of the M symbols is equally likely to be transmitted and when an amplitude or timing error occurs, either of the two "adjacent" symbols is equally likely to be selected. Neglecting "end effects," it can be shown that the average number of bit errors is approximately

$$E_{AV} = \frac{2^B - 1}{2^{B-1}}$$

when the natural ordering of the binary numbers is used. Thus, for large B, the average number of errors due to amplitude or timing errors is approximately two, or double the number of errors that occur if the Gray code is used. For example, when B=8, the average number of errors is 1.992.

The Gray code can be generated by using modulo-2 arithmetic. Let

$$X = (x_1, x_2, \cdots, x_B)$$

be the standard binary representation. The Gray code representation is given by

$$G = (g_1, g_2, \cdots, g_B),$$

with

$$g_1 = x_1$$
$$g_k = x_{k-1} \oplus x_k, k = 2,3,\ldots,B$$

The symbol $\oplus$ denotes addition modulo-2. This method of generating the Gray code is time consuming because it requires converting the integers 0, 1,..., 2^B-1 from decimal to binary, examining $B\times2^B$ bits and performing $(B-1)\times2^B$ modulo-2 additions. However, generating the Gray code can be accomplished quickly and simply by using the recursive method described below, which is applied to the decimal representation of the integers.

Figure 1 shows the decimal representations of the Gray code as a function of the naturally ordered binary numbers for B=4 and B=5. From this figure we observe the following: The first half for B=5 is the Gray code for B=4, and the second half is the reverse of the Gray code for B=4 with 16 added. This observation leads to the following recursive method. Start with the vector $y_1 = (0,1)$ (which is the Gray code for one bit, in decimal). Let

$$y_m = (y_{m-1}, y_{m-1}^R + 2^{m-1}),\ m = 2, 3, \ldots, B,$$

where y_m^R is y_m in reverse order. The vector y_B, of length $M=2^B$, has as its elements the decimal representation of the B-bit Gray code. This recursive computation can be accomplished easily by using the following MATLAB[R] program.

```
function y=graycode(B)
y=[0 1];
for k=2:B;
        y=[y y(end:-1:1)+2^(k-1)];
end;
```

Note that each pass through the loop in the above program produces the k-bit Gray code. The MATLAB[R] function z=dec2bin(y) can be used to obtain the binary representation of the Gray code.

In Figure 2 are normalized plots of the decimal representation of the Gray code versus the naturally ordered binary numbers for B=15 and B=16. This further illustrates the recursive algorithm and the fractal nature of the curves that result.

Reference

1. Proakis, J.G., *Digital Communications*, Third Edition, McGraw-Hill, New York, 1995.

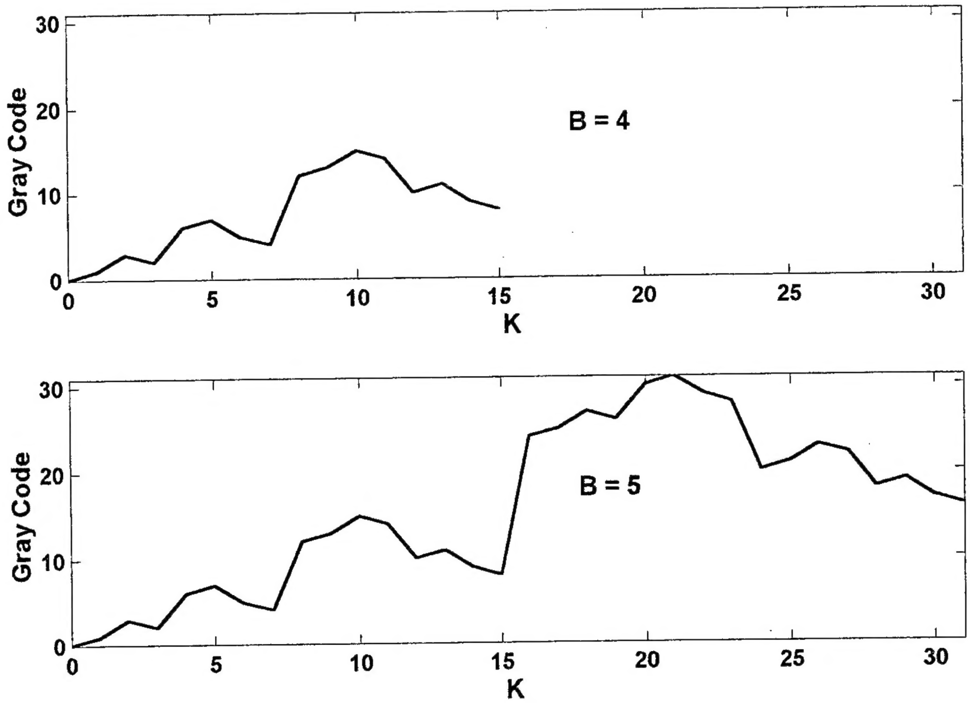

Figure 1. Gray code in decimal for B=4 and B=5.

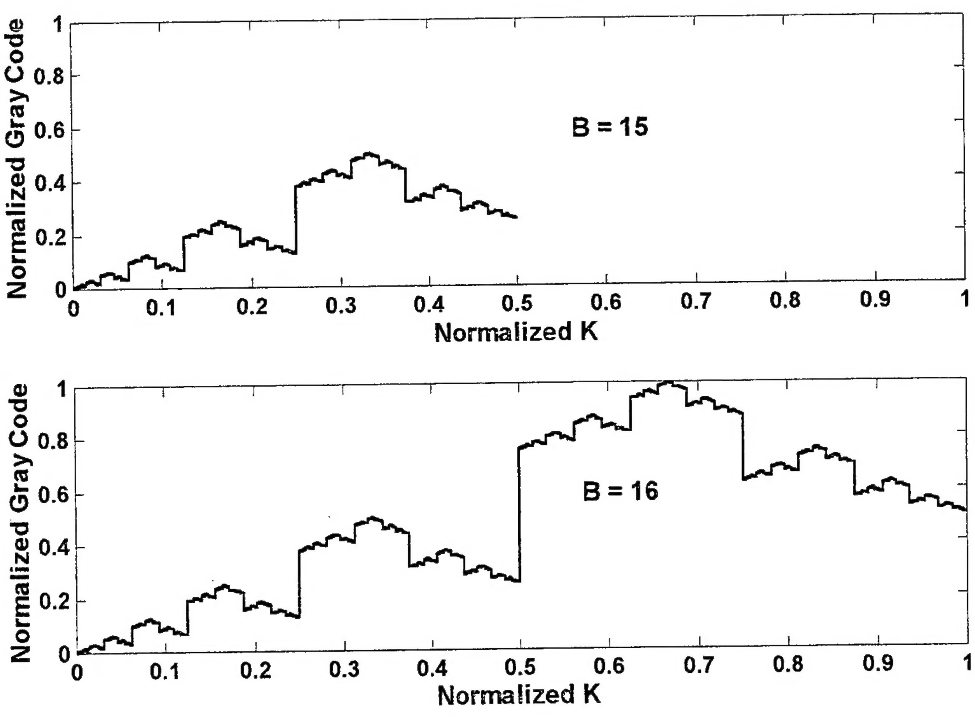

Figure 2. Normalized Gray code in decimal for B=15 and B=16.